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LETTER TO THE EDITOR

Magnetisation profiles for anisotropic spin glasses

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Abstract. The thermodynamic (field cooled) magnetisation is analysed throughout the spin glass domain (transverse T, longitudinal L or mixed LT phases) for a class of anisotropic m vector models, in the limit of a small collinear field. In zero field the field cooled susceptibility χ_T (χ_L) is independent of temperature throughout the T (L and LT) phases. In a finite collinear field we observe the celebrated Parisi '$h^{4/3}$' dependence of the field cooled magnetisation throughout the L and LT phases. For the T phase no simple characterisation is apparent.

The Parisi solution (1979, 1980) of the Sherrington–Kirkpatrick m vector spin glass described by Elderfield and Sherrington (1982a, b) and Gabay *et al* (1982) and recently interpreted by Parisi (1983) and de Dominicis and Young (1983) in terms of the metastable states of Thouless *et al* (1977) and Bray and Moore (1981) is now widely accepted. A characteristic feature of the Ising (or $m = 1$) solution is that in the spin glass phase the field cooled magnetisation profile is flat or independent of temperature (Sompolinsky 1981), at least for small external fields (Elderfield 1983). For the isotropic m vector model in a finite field we observe (Elderfield and Sherrington 1982c) that the profile is flat only for temperatures substantially lower than that associated with the onset of transverse spin glass ordering ($m > 1$), whilst for an anisotropic m vector model preliminary estimates (Elderfield and Sherrington 1982a) suggest that flat magnetisation profiles (transverse/longitudinal) are associated with (transverse/longitudinal) or mixed spin glass phases in the limit of vanishingly small external fields. At present, however, the structure of the profile is known precisely only in the vicinity of the multicritical point associated with the simultaneous appearance of both transverse and longitudinal freezing, i.e. weak anisotropy and small external fields. To remedy this shortcoming, we shall here derive exact relations, closely associated with those obtained by Sommers (1983) for the Ising problem, which in the limit of weak external fields allows us to characterise the profiles deep in the spin glass phases. In this communication we shall not consider the zero field cooled or quasi-equilibrium magnetisation which is considerably harder to determine.

A simple model with an interesting phase diagram is that of a spin glass with uniaxial anisotropy. To facilitate the construction of a mean field description we follow Sherrington and Kirkpatrick (1975) and adopt a Hamiltonian

$$\mathcal{H} = -\sum_{i,j=1}^{N}\left(J_{ij}^{\mathrm{L}}\sum_{\mu,\nu=1}^{m} d_\mu d_\nu S_{\mu i}S_{\nu j} + J_{ij}^{\mathrm{T}}\sum_{\mu,\nu,\phi=1}^{m} t_{\phi\mu}t_{\phi\nu}S_{\mu i}S_{\nu j}\right) - \sum_{i=1}^{N} D\left(\sum_{\mu,\nu=1}^{m} d_\mu d_\nu S_{\mu i}S_{\nu j}\right) - \sum_{i=1}^{N} h\left(\sum_{\mu=1}^{m} d_\mu S_{\mu i}\right) \qquad (1)$$

which describes the interaction of m-dimensional classical vectors ($|S|^2 \equiv \Sigma_{\mu=1}^{m} (S_\mu)^2 = m$) coupled via a set of infinite range exchanges $\{J_{ij}^{\mathrm{T}}\}$, $\{J_{ij}^{\mathrm{L}}\}$ with distributions

$$P(J_{ij}^A) = (N/2\pi J_A^2)^{1/2} \exp[-N(J_{ij}^A)^2/2J_A^2] \tag{2}$$

for $A = \mathrm{L}$, T. The anisotropy axis is defined by the unit vector $\boldsymbol{d}$ with the projector $t_{\mu\nu} \equiv \delta_{\mu\nu} - d_\mu d_\nu$ onto the transverse degrees of freedom. In the absence of exchange anisotropy ($J_{\mathrm{T}} = J_{\mathrm{L}}$) the model reduces to that considered by Cragg and Sherrington (1982), Roberts and Bray (1982), for vanishing single ion anisotropy ($D = 0$) the system has been discussed by Bray and Viana (1983), whilst the isotropic case ($D = 0$, $J_{\mathrm{T}} = J_{\mathrm{L}}$ but $h \neq 0$) was originally introduced by Gabay and Toulouse (1981). To simplify considerations we shall consider only the case for which the external field $\boldsymbol{h}$ is collinear with the anisotropy axis $\boldsymbol{d}$.

Using the replica trick, we obtain the free energy per spin in the thermodynamic limit in the form of the extremal problem

$$\beta F = \lim_{n \to 0} [\mathrm{ext}(\beta F(\{q^{\alpha\beta}, p^{\alpha\beta}\}, x))] \tag{3}$$

where the free energy functional is of the form

$$\begin{aligned} n\beta F(\{q^{\alpha\beta}, p^{\alpha\beta}\}, x) &= \tfrac{1}{4}\beta^2 \sum_{(\alpha\beta)}^{n} (J_{\mathrm{T}}^2 (m-1)(q^{\alpha\beta})^2 + J_{\mathrm{L}}^2 (p^{\alpha\beta})^2) \\ &+ \tfrac{1}{4} n\beta^2 \{J_{\mathrm{T}}^2 (m-1)(1-x)^2 + J_{\mathrm{L}}^2 [1 + (m-1)x]^2 - 2mJ_{\mathrm{T}}^2 (1-x)\} \\ &- \log\left\{ \mathop{\mathrm{Tr}}_{\{S^\alpha\}} \exp\left[\tfrac{1}{2}\beta^2 \left(\sum_{(\alpha\beta)}^{n} J_{\mathrm{T}}^2 q^{\alpha\beta} t_{\mu\nu} S_\mu^\alpha S_\nu^\beta + J_{\mathrm{L}}^2 p^{\alpha\beta} d_\mu d_\nu S_\mu^\alpha S_\nu^\beta \right) \right.\right. \\ &- \tfrac{1}{2}\beta^2 \left(\sum_{\alpha}^{n} [J_{\mathrm{T}}^2 (1-x) - J_{\mathrm{L}}^2 (1 + (m-1)x)] d_\mu d_\nu S_\mu^\alpha S_\nu^\alpha \right) \\ &\left.\left. + \beta \left(\sum_{\alpha}^{n} D d_\mu d_\nu S_\mu^\alpha S_\nu^\alpha \right) + \beta \left(\sum_{\alpha}^{n} h d_\mu S_\mu^\alpha \right) \right] \right\}. \end{aligned} \tag{4}$$

Here (α, β) refers to a sum over distinct pairs and we have adopted the conventional repeated index notation. As usual the order parameters $q^{\alpha\beta}$, $p^{\alpha\beta}$, x are associated respectively with transverse spin glass, longitudinal spin glass and quadrupolar ordering through the familiar extremal equations

$$\begin{aligned} \partial(n\beta F)/\partial q^{\alpha\beta} &= \tfrac{1}{2}(\beta J_{\mathrm{T}})^2 [(m-1)q^{\alpha\beta} - t_{\mu\nu} \langle S_\mu^\alpha S_\nu^\beta \rangle] = 0 \\ \partial(n\beta F)/\partial p^{\alpha\beta} &= \tfrac{1}{2}(\beta J_{\mathrm{L}})^2 [p^{\alpha\beta} - d_\mu d_\nu \langle S_\mu^\alpha S_\nu^\beta \rangle] = 0 \\ \partial(n\beta F)/\partial x &= \tfrac{1}{2} n [(m-1)(\beta J_{\mathrm{L}})^2 + (\beta J_{\mathrm{T}})^2][1 + (m-1)x - d_\mu d_\nu \langle S_\mu^\alpha S_\nu^\alpha \rangle] = 0. \end{aligned} \tag{5}$$

The restriction to collinear fields and zero mean exchange allows us to disregard the possibility of spin glass ordering transverse to both $\boldsymbol{h}$ and $\boldsymbol{d}$.

To simplify (4), (5) it is useful for our purposes to adopt a variation of the Parisi ansatz due to Sompolinsky (1981), and de Dominicis *et al* (1982) which leads to the representation

$$\beta F_{\mathrm{S}} = \mathrm{ext}(\beta F_{\mathrm{S}}(\{q(r), \Delta_q(r), p(r), \Delta_p(r)\}, x))_{\Delta_p(1) = \Delta_q(1) = 0} \tag{6}$$

in terms of order parameter functions $q(r)$, $p(r)$, $\Delta_q(r)$, $\Delta_p(r)$ defined on the interval $(0, 1)$. The free energy functional is defined as

$$-\beta F_S(\{q(r), \Delta_q(r), p(r), \Delta_p(r)\}, x)$$
$$= \tfrac{1}{4}(\beta J_T)^2\left((m-1)q^2(1) - 2mq(1) + 2(m-1)\int_0^1 dr\, \Delta_q'(r)q(r)\right)$$
$$+ \tfrac{1}{4}(\beta J_L)^2\left(p^2(1) + 2\int_0^1 dr\, \Delta_p'(r)p(r)\right) - \tfrac{1}{4}\{(\beta J_L)^2[1+(m-1)x]^2$$
$$+ (\beta J_T)^2(1-x)^2 - 2m(\beta J_T)^2(1-x)\} + \overline{\log(Z^*(H^*, D^*))}_{h=hd}$$
$$+ \tfrac{1}{2}\beta^2 \int_0^1 dr(J_T^2\, \Delta_q'(r)t_{\mu\nu} + J_L^2\, \Delta_p'(r)d_\mu d_\nu)\overline{[m_\mu]_r[m_\nu]_r} \qquad (7)$$

in terms of the partition function Z^*

$$Z^* = \operatorname*{Tr}_{\{s\}} \exp\{\beta[H^* \cdot S + D^*(S \cdot d)^2]\} \qquad (8)$$

where H^* is an effective field

$$\beta H_\nu^* = \beta h_\nu + (\beta J_T)t_{\mu\nu}\left(z_\mu\sqrt{q(0)} + \int_0^1 dr(z_\mu(r)\sqrt{q'(r)} - (\beta J_T)[m_\mu]_r\Delta_q'(r))\right)$$
$$+ (\beta J_L)d_\mu d_\nu\left(z_\mu\sqrt{p(0)} + \int_0^1 dr(z_\mu(r)\sqrt{p'(r)} - (\beta J_L)[m_\mu]_r\Delta_p'(r))\right) \qquad (9)$$

and D^* is of the form

$$\beta D^* = \beta D + \tfrac{1}{2}(\beta J_L)^2[1+(m-1)x - p(1)] + \tfrac{1}{2}(\beta J_T)^2(1-x+q(1)). \qquad (10)$$

Here $\boldsymbol{m}$ is the magnetisation associated with (8), in the collinear limit.

$$m = \partial(\ln Z^*)/\partial(\beta h_\mu)|_{h=hd}. \qquad (11)$$

Finally the bar $\overline{(\ldots)}$ denotes averaging over the gaussian random vectors $z(r)$, $r \in (0, 1)$ and z for which

$$\overline{z_\mu(r)} = 0 = \overline{z_\mu} \qquad \overline{z_\mu(r)z_\nu(r')} = \delta_{\mu\nu}\delta(r-r') \qquad \overline{z_\mu z_\nu} = \delta_{\mu\nu} \qquad (12)$$

whilst $[\ldots]_s$ defines a restricted average over the variables $z(r)$, $r > s$.

The extrema equations associated with (6) *et seq* are of the Sompolinsky form

$$-\partial(\beta F_S)/\partial q'(r) = \tfrac{1}{2}(\beta J_T)^2(m-1)$$
$$\times[q(1) - 1 + x - \Delta_q(r) + [1/(m-1)]t_{\mu\nu}\overline{(\partial m_\mu/\partial(\beta h_\nu(r)))}] = 0 \qquad (13)$$
$$-\partial(\beta F_S)/\partial p'(r) = \tfrac{1}{2}(\beta J_L)^2[p(1) - 1 - (m-1)x - \Delta_p(r) + d_\mu d_\nu\overline{(\partial m_\mu/\partial(\beta h_\nu(r)))}] = 0 \qquad (14)$$
$$-\partial(\beta F_S)/\partial\Delta_q'(r) = \tfrac{1}{2}(\beta J_T)^2[(m-1)q(r) - t_{\mu\nu}\overline{[m_\mu]_r[m_\nu]_r}] = 0 \qquad (15)$$
$$-\partial(\beta F_S)/\partial\Delta_p'(r) = \tfrac{1}{2}(\beta J_L)^2[p(r) - d_\mu d_\nu\overline{[m_\mu]_r[m_\nu]_r}] = 0 \qquad (16)$$
$$\partial(\beta F_S)/\partial x = \tfrac{1}{2}[(\beta J_L)^2(m-1) + (\beta J_T)^2][1 + (m-1)x - \overline{\partial(\log Z^*)/\partial(\beta D)})] = 0 \qquad (17)$$

where $\beta h_\nu(r) \equiv \beta(J_T\sqrt{q'}(r)t_{\mu\nu} + J_L\sqrt{p'}(r)d_\mu d_\nu)z_\mu(r))$. Variation with respect to $q(0)$ ($p(0)$) simply reproduces (13) ((14)) in the limit $r \to 0$. The relations (13), (14) are directly useful for, following Parisi/Sompolinsky, we can identify the field cooled *local* susceptibility

$$\chi^{l}_{\mu\nu}(\mathrm{FC}) \equiv \overline{\frac{\partial m_\mu}{\partial h_\nu}} = \begin{cases} (\beta/N) \sum_{i=1}^{N} (\langle S_{\mu i} S_{\nu i}\rangle - \langle S_{\mu i}\rangle\langle S_{\nu i}\rangle) \\ \{d_\mu d_\nu \beta[1+(m-1)x - p(1) + \Delta_p(0)] + t_{\mu\nu}\beta(1-x-q(1)+\Delta_q(0))\} \end{cases} \tag{18}$$

and the zero field cooled *local* susceptibility

$$\chi^{l}_{\mu\nu}(\mathrm{ZFC}) \equiv \frac{\partial m_\mu}{\partial h_\nu(1)} = \begin{cases} (\beta/N) \sum_{i=1}^{N} (\langle S_{\mu i} S_{\nu i}\rangle_{\mathrm{R}} - \langle S_{\mu i}\rangle_{\mathrm{R}}\langle S_{\nu i}\rangle_{\mathrm{R}}) \\ \beta\{d_\mu d_\nu [1+(m-1)x - p(1)] + t_{\mu\nu}(1-x-q(1)]\}. \end{cases} \tag{19}$$

Here $\langle\ldots\rangle$ denotes a full Gibbs average whilst $\langle\ldots\rangle_{\mathrm{R}}$ denotes the restricted Gibbs average appropriate to the statistical mechanics of an isolated metastable state (Parisi 1983, de Dominicis and Young 1983). It is of course important to realise that the global susceptibility

$$\chi_{\mu\nu}(\mathrm{FC}) = \overline{\frac{\mathrm{d}m_\mu}{\mathrm{d}h_\nu}} = (\beta/N) \sum_{i,j=1}^{N} (\langle S_{\mu i} S_{\nu j}\rangle - \langle S_{\mu i}\rangle\langle S_{\nu j}\rangle) \tag{20}$$

is generally distinct from $\chi^{l}_{\mu\nu}(\mathrm{FC})$, the limiting case of zero field being exceptional. We have used a total derivative in (20) to emphasise that $\chi_{\mu\nu}(\mathrm{FC})$ includes contributions arising from the explicit $\boldsymbol{h}$ dependence of the order parameters $q(r)$, $p(r)$, $\Delta_q(r)$, $\Delta_p(r)$, x. For the global zero field cooled susceptibility we may similarly write

$$\chi_{\mu\nu}(\mathrm{ZFC}) = (\beta/N) \sum_{i,j=1}^{N} \langle S_{\mu i} S_{\nu j}\rangle_{\mathrm{R}} - \langle S_{\mu i}\rangle_{\mathrm{R}}\langle S_{\nu j}\rangle_{\mathrm{R}}; \tag{21}$$

however, for this case there is at present no simple operational prescription for computing (21) within the current framework.

To date, the discussion of Heisenberg spin glasses has revolved around the Parisi analysis of Elderfield and Sherrington (1982a) and Gabay *et al* (1982) which corresponds to the 'gauge' choice

$$\Delta'_a(r) = -ra'(r), \qquad a = p, q, \tag{22}$$

on the Sompolinsky functional F_{S}, i.e.

$$F_{\mathrm{P}}(\{q(r), p(r)\}, x) = (F_{\mathrm{S}}(\{q(r), \Delta_q(r), p(r), \Delta_p(r)\}, x))_{\Delta'_a(r)=-ra'(r), a=p,q}. \tag{23}$$

This relation follows directly from the observation that F_{S} may be rewritten in the form

$$\begin{aligned} -F_{\mathrm{S}} = {} & \tfrac{1}{4}(\beta J_{\mathrm{T}})^2\left((m-1)q^2(1) - 2mq(1) + 2(m-1)\int_0^1 \mathrm{d}r\ \Delta'_q(r)q(r)\right) \\ & + \tfrac{1}{4}(\beta J_{\mathrm{L}})^2\left(p^2(1) + 2\int_0^1 \mathrm{d}r\ \Delta'_p(r)p(r)\right) \\ & - \tfrac{1}{4}\{(\beta J_{\mathrm{L}})^2[1+(m-1)x]^2 + (\beta J_{\mathrm{T}})^2(1-x)^2 - 2m(\beta J_{\mathrm{T}})^2(1-x)\} \\ & + \overline{f(\{y_\nu = hd_\nu + (J_{\mathrm{T}} t_{\mu\nu}\sqrt{q}(0) + J_{\mathrm{L}} d_\mu d_\nu \sqrt{p}(0))z_\mu\}, 0)}, \end{aligned} \tag{24}$$

where the function $f(y, r)$ satisfies the Parisi-like equation

$$-(\partial/\partial r)(f(y,r)) = \tfrac{1}{2}\{[(\beta J_{\mathrm{T}})^2 q'(r) t_{\mu\nu} + (\beta J_{\mathrm{L}})^2 p'(r) d_\mu d_\nu]\, \partial^2 f(y,r)/\partial y_\mu\, \partial y_\nu$$
$$-[((\beta J_{\mathrm{T}})^2 \Delta_q'(r) t_{\mu\nu} + (\beta J_{\mathrm{L}})^2 \Delta_p'(r) d_\mu d_\nu)](\partial f(y,r)/\partial y_\nu)(\partial f(y,r)/\partial y_\mu)\} \qquad (25)$$

with the boundary condition

$$f(y, 1) = \log\left[\operatorname*{Tr}_{S}(\exp\{\beta[y \cdot S + D^*(d \cdot S)^2]\})\right] \qquad (26)$$

(see Sommers (1983) for a discussion of the Ising ($m = 1$) case). As in the Ising case, the Sompolinsky relations (13)–(16) are in a sense incomplete, for (24) and (25) show directly that the free energy functional F_{S} is invariant under arbitrary reparametrisations $r \to g(r)$, g monotonic. Consequently this approach makes absolute predictions only for the behaviour at the endpoints of the natural interval chosen as (0, 1), which correspond to the physically important FC($r = 0$) and ZFC($r = 1$) limits, as outlined above. In the present context the Sompolinsky equations are simpler to handle and lead to identical predictions to those of the Parisi (or 'gauge fixed') approach (22). Our analysis will, therefore, be based on a discussion of the properties of F_{S}, which if desired may easily be generalised to describe the Parisi case.

In order to obtain the field cooled magnetisation $\boldsymbol{M}$

$$\boldsymbol{M} = \overline{\boldsymbol{m}} \qquad (27)$$

or the associated global susceptibility

$$\chi_{\mu\nu}(\mathrm{FC}) \equiv \chi_{\mathrm{L}}(\mathrm{FC}) d_\mu d_\nu + \chi_{\mathrm{T}}(\mathrm{FC}) t_{\mu\nu} \equiv \mathrm{d}M_\mu/\mathrm{d}h_\nu = \overline{\mathrm{d}m_\mu/\mathrm{d}h_\nu} \qquad (28)$$

it is helpful to follow Sommers (1983) and derive a series of useful identities. Differentiating first the extremal relations (15), (16) with respect to r gives the identity

$$\begin{pmatrix} q'(r) \\ p'(r) \end{pmatrix} = \overline{\left[\frac{\partial m_\mu}{\partial(\beta h_\theta(r))}\right]_r \left[\frac{\partial m_\nu}{\partial(\beta h_\phi(r))}\right]_r}$$
$$\times \begin{pmatrix} [1/(m-1)](\beta J_{\mathrm{T}})^2 t_{\mu\nu} t_{\theta\phi} & [1/(m-1)](\beta J_{\mathrm{L}})^2 t_{\mu\nu} d_\theta d_\phi \\ (\beta J_{\mathrm{T}})^2 d_\mu d_\nu t_{\theta\phi} & (\beta J_{\mathrm{L}})^2 d_\mu d_\nu d_\theta d_\phi \end{pmatrix} \begin{pmatrix} q'(r) \\ p'(r) \end{pmatrix}. \qquad (29)$$

To derive an analogous expression relating Δ_p', Δ_q' we observe from (7) that

$$\frac{\partial m_\mu}{\partial(\beta h_\nu(r))} = \frac{\partial m_\mu}{\partial(\beta h_\nu)} - \int_r^1 \mathrm{d}s\, (t_{\theta\phi}(\beta J_{\mathrm{T}})^2 \Delta_q'(s) + d_\theta d_\phi (\beta J_{\mathrm{L}})^2 \Delta_p'(s)) \left(\frac{\partial m_\mu}{\partial(\beta h_\theta)}\right) \left[\frac{\partial m_\phi}{\partial(\beta h_\nu(r))}\right]_s, \qquad (30)$$

which on differentiation with respect to r leads to the similar relation

$$\frac{\partial}{\partial r}\left(\frac{\partial m_\mu}{\partial(\beta h_\nu(r))}\right) = (t_{\theta\phi}(\beta J_{\mathrm{T}})^2 \Delta_q'(r) + d_\theta d_\phi (\beta J_{\mathrm{L}})^2 \Delta_p'(r)) \left(\frac{\partial m_\mu}{\partial(\beta h_\theta)}\right) \left[\frac{\partial m_\phi}{\partial(\beta h_\nu(r))}\right]_r$$
$$- \int_r^1 \mathrm{d}s\, (t_{\theta\phi}(\beta J_{\mathrm{T}})^2 \Delta_q'(s) + d_\theta d_\phi (\beta J_{\mathrm{L}})^2 \Delta_p'(s)) \left(\frac{\partial m_\mu}{\partial(\beta h_\theta)}\right) \frac{\partial}{\partial r}\left[\frac{\partial m_\phi}{\partial(\beta h_\nu(r))}\right]_s. \qquad (31)$$

Comparing (30), (31) we now find directly the identity

$$(\partial/\partial r)(\partial m_\mu/\partial(\beta h_\nu(r)))$$
$$= (t_{\theta\phi}(\beta J_{\mathrm{T}})^2 \Delta_q'(r) + d_\theta d_\phi (\beta J_{\mathrm{L}})^2 \Delta_p'(r))[\partial m_\phi/\partial(\beta h_\nu(r))]_r (\partial m_\mu/\partial(\beta h_\theta)), \qquad (32)$$

which allows us to evaluate the derivative with respect to r of the extremal relations (13), (14) in the desired form:

$$\begin{pmatrix}\Delta_q'(r)\\ \Delta_p'(r)\end{pmatrix} = \overline{\left[\frac{\partial m_\mu}{\partial(\beta h_\theta(r))}\right]_r \left[\frac{\partial m_\theta}{\partial(\beta h_\nu(r))}\right]_r}$$

$$\times \begin{pmatrix}[1/(m-1)](\beta J_{\mathrm{T}})^2 t_{\mu\nu} t_{\theta\phi} & [1/(m-1)](\beta J_{\mathrm{L}})^2 t_{\mu\nu} d_\theta d_\phi \\ (\beta J_{\mathrm{T}})^2 d_\mu d_\nu t_{\theta\phi} & (\beta J_{\mathrm{L}})^2 d_\mu d_\nu d_\theta d_\phi\end{pmatrix}\begin{pmatrix}\Delta_q'(r)\\ \Delta_p'(r)\end{pmatrix}. \tag{33}$$

Notice that unlike the Ising case ($m = 1$) Sommers (1983) the relations (29), (33) are not generally equivalent. The final relation we shall require is

$$\frac{\partial}{\partial r}\left(\overline{\left[\frac{\partial m_\mu}{\partial(\beta h_\theta(r))}\right]_r \left[\frac{\partial m_\nu}{\partial(\beta h_\phi(r))}\right]_r}\right)$$

$$= [t_{\lambda\delta}(\beta J_{\mathrm{T}})^2 q'(r) + d_\lambda d_\delta(\beta J_{\mathrm{L}})^2 p'(r)]$$

$$\times \left(\overline{\left[\frac{\partial^2 m_\mu}{\partial(\beta h_\theta(r))\,\partial(\beta h_\lambda(r))}\right]_r \left[\frac{\partial^2 m_\nu}{\partial(\beta h_\phi(r))\,\partial(\beta h_\delta(r))}\right]_r}\right)$$

$$+ [t_{\lambda\delta}(\beta J_{\mathrm{T}})^2 \Delta_q'(r) + d_\lambda d_\delta(\beta J_{\mathrm{L}})^2 \Delta_p'(r)]$$

$$\times (\overline{[\partial m_\mu/\partial(\beta h_\theta(r))]_r [\partial m_\nu/\partial(\beta h_\lambda(r))]_r [\partial m_\delta/\partial(\beta h_\phi(r))]} + \text{perm}) \tag{34}$$

where the first term follows in analogy with (29) and the second with (32).

Broadly the field cooled susceptibility $\chi_{\mu\nu}$(FC) depends only on the nature of the spin glass phase (transverse, longitudinal or mixed) in the FC regime $r \to 0$, at least if the external field is 'weak'. Consider for example the simplest isotropic model ($J_{\mathrm{T}} = J_{\mathrm{L}} = J$, $D = 0$ and $\boldsymbol{h} = \boldsymbol{0}$) which is known to exhibit an isotropic spin glass phase at temperatures $T \leqslant J$. In the spin glass phase we expect both $q(r) = p(r)$, $-\Delta_q(r) = -\Delta_p(r)$ to be monotonic increasing functions on an interval I (chosen as (0, 1) by means of the gauge transformation), which quite generally (in zero external field) satisfy the boundary condition

$$q(0) = p(0) = 0 \tag{35}$$

(see Sommers (1983) for a discussion of the Ising case). Now for such a solution we observe that in the limit $r \to 0$, the averages in (29), (33) trivialise leading to the relations

$$J\chi(\mathrm{FC}) = 1$$

or

$$q'(0) = p'(0) = \Delta_q'(0) = \Delta_p'(0) = 0 \tag{36}$$

where we have identified the isotropic field cooled susceptibility χ(FC)

$$\chi_{\mu\nu}(\mathrm{FC}) \equiv \delta_{\mu\nu}\chi(\mathrm{FC}) \equiv \mathrm{d}M_\mu/\mathrm{d}h_\nu = \overline{\mathrm{d}m_\mu/\mathrm{d}h_\nu}. \tag{37}$$

Naturally in zero field the global and local susceptibilities defined above are indistinguishable, for the gauge symmetry $J_{ij} \to -J_{ij}$, $S_i \to -S_i$ any i, all $j = 1, \ldots, N$ ensures that off diagonal contributions vanish identically (see (18), (20)). For the isotropic case we therefore see that *throughout* the spin glass phase the susceptibility χ(FC) is independent of temperature

$$\chi(\mathrm{FC}) = (1/J) \qquad \text{isotropic spin glass} \tag{38}$$

leading in 'weak' external fields to the flat magnetisation profile

$$m_\mu = \lambda_{\mu\nu}(\mathrm{FC})h_\nu = (1/J)h_\mu \qquad (39)$$

first observed by Parisi (1980) for the Ising case. The finite field corrections to (39) will be ignored for the present.

Consider now the more interesting anisotropic case $D \neq 0$, $J_T \neq J_L$ but $\boldsymbol{h} = 0$ for which the phase diagrams are of the form sketched in figure 1, as shown by Cragg and Sherrington (1982), Roberts and Bray (1982), Bray and Viana (1983)†. Briefly one identifies domains of transverse T, longitudinal L and mixed LT spin glass ordering.

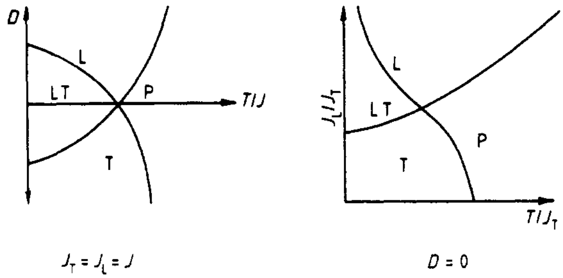

Figure 1. Phase diagrams for the anisotropic case $D \neq 0$, $J_T = J_L$ or $D = 0$, $J_T \neq J_L$ both for $\boldsymbol{h} = 0$.

As one might expect (Elderfield and Sherrington 1982a) the L, T phases are associated with pure Ising and $(m-1)$ isotropic spin glass ordering respectively. In the T phase we have $\Delta_p(r) = p(r) = 0$ whilst $q(r)$, $-\Delta_q(r)$ are monotonic increasing functions with the boundary condition $q(0) = 0$ (in zero transverse field). In the limit $r \to 0$ (29), (33) and (28) imply

$$J_T \chi_T(\mathrm{FC}) = 1$$

or

$$q'(0) = \Delta_q'(0) = 0 \qquad (40)$$

leading to the observation that the *transverse* field cooled susceptibility $\chi_T(\mathrm{FC})$ is independent of temperature in a *transverse* spin glass phase:

$$\chi_T(\mathrm{FC}) = (1/J_T) \qquad \text{transverse spin glass.} \qquad (41)$$

Similarly in the *longitudinal* L spin glass phase the *longitudinal* field cooled susceptibility $\chi_L(\mathrm{FC})$ is flat

$$\chi_L(\mathrm{FC}) = (1/J_L) \qquad \text{longitudinal spin glass.} \qquad (42)$$

For the mixed phase LT the analysis is more complicated, as one would expect, for now both $(q(r), -\Delta_q(r))$ and $(p(r), -\Delta_p(r))$ are monotonic increasing functions with the boundary condition $q(0) = p(0) = 0$ (in zero external field), coupled through the non-trivial relations (29), (33). Analysing the $r \to 0$ limit as before we obtain the

† The paper of Bray and Viana neatly avoids the problems associated with replica symmetry breaking effects by using the known internal field distributions to calculate limits of stability at $T = 0$.

expressions

$$\begin{pmatrix} q'(r) \\ \Delta_q'(r) \end{pmatrix} = (J_T\chi_T(\mathrm{FC}))^2 \begin{pmatrix} q'(0) \\ \Delta_q'(0) \end{pmatrix} + r \begin{pmatrix} q''(0) \\ \Delta_q''(0) \end{pmatrix} \tag{43}$$

$$\begin{pmatrix} p'(r) \\ \Delta_p'(r) \end{pmatrix} = (J_L\chi_L(\mathrm{FC}))^2 \begin{pmatrix} p'(0) \\ \Delta_p'(0) \end{pmatrix} + r \begin{pmatrix} p''(0) \\ \Delta_p''(0) \end{pmatrix} \tag{44}$$

up to corrections of order r^2. In the *mixed phase* we therefore find that both the *transverse and longitudinal* field cooled susceptibilities are flat†:

$$\chi_L(\mathrm{FC}) = (1/J_L) \qquad \chi_T(\mathrm{FC}) = (1/J_T) \qquad \text{mixed spin glass.} \tag{45}$$

Introducing a finite collinear field $\boldsymbol{h} = h\boldsymbol{d}$ we would anticipate, following Elderfield and Sherrington (1982b, c) and Gabay *et al* (1982), some interesting new effects. For the Ising-like L phase $\Delta_q(r)$, $q(r)$ remain zero whilst $p(r)$, $-\Delta_p(r)$, still monotonic increasing functions, now no longer obey the boundary condition $p(0) = 0$, which is appropriate only in the zero field case. In the $r \to 0$ limit the averages in (29), (33) are now non-trivial, giving relations of the form

$$1 = (\beta J_L)^2 \overline{(\partial^2 f/\partial(\beta h))^2}$$

or

$$p'(0) = \Delta_p'(0) = 0 \tag{46}$$

where the function $f((h + J_L\sqrt{p(0)}z)\boldsymbol{d}, 0)$ is defined by (25), (26). Similarly (15), (27), (28) imply

$$\boldsymbol{M} \equiv M\boldsymbol{d} = \overline{\partial f/\partial(\beta h)\, \boldsymbol{d}} \qquad p(0) = \overline{(\partial f/\partial(\beta h))^2} \qquad \chi_L^l = \beta\overline{(\partial^2 f/\partial(\beta h)^2)}. \tag{47}$$

In this Ising-like phase we expect $p(0) \sim h^{2/3}$ so developing f systematically as a function of $(h + J_L\sqrt{p(0)}z)$ and performing the trivial gaussian averages we find

$$M/h = 1 - \tfrac{3}{4}(h^2/p(0)) + \dots \qquad \text{(longitudinal spin glass)} \tag{48}$$

where $p(0)$ is given by the relation

$$h^2 = \tfrac{1}{3}(\partial^4 f(y, 0)/\partial(\beta y)^4|_{y=0})^2(\beta^2 p(0))^3 + \dots. \tag{49}$$

To leading order in h we therefore observe the celebrated Parisi '$h^{4/3}$' dependence of the magnetisation *throughout* the L phase. It is also interesting to note that the local field cooled susceptibility $\chi_L^l(\mathrm{FC})$ is certainly different from global susceptibility $\chi_L(\mathrm{FC})$ in a finite field as suggested above.

$$\chi_L^l = M/h + \dots \neq \chi_L = \mathrm{d}M/\mathrm{d}h. \tag{50}$$

In the transverse T or mixed LT spin glass phases the analysis is more complicated. For in these cases *both* $(q(r), -\Delta_q(r))$ and $(p(r), -\Delta_p(r))$ are monotonic increasing functions (Elderfield and Sherrington 1982b) satisfying the non-trivial boundary condition $q(0) = 0$ but $p(0) \neq 0$, reflecting the presence of the collinear field. As above the averages in (29), (33) are now non-trivial in the limit $r \to 0$, giving the relations

$$\begin{pmatrix} q'(r) \\ \Delta_q'(r) \end{pmatrix} = t_{\mu\nu}t_{\theta\phi}(\beta J_T)^2 \left(\frac{\partial^2 f}{\partial(\beta h_\mu)\,\partial(\beta h_\theta)}\right)\left(\frac{\partial^2 f}{\partial(\beta h_\nu)\,\partial(\beta h_\phi)}\right) \begin{pmatrix} q'(0) \\ \Delta_q'(0) \end{pmatrix} + r \begin{pmatrix} q''(0) \\ \Delta_q''(0) \end{pmatrix} \tag{51}$$

† In Elderfield and Sherrington (1982a) we presented an alternative solution, which is now known to be unstable.

$$\begin{pmatrix} p'(r) \\ \Delta_p'(r) \end{pmatrix} = d_\mu d_\nu d_\theta d_\phi (\beta J_{\rm L})^2 \left(\frac{\partial^2 f}{\partial(\beta h_\mu)\,\partial(\beta h_\theta)}\right)\left(\frac{\partial^2 f}{\partial(\beta h_\nu)\,\partial(\beta h_\phi)}\right)\begin{pmatrix} p'(0) \\ \Delta_p'(0) \end{pmatrix} + r\begin{pmatrix} p''(0) \\ \Delta_p''(0) \end{pmatrix} \tag{52}$$

up to terms of order r^2. Throughout the T phase we must, therefore, conclude in agreement with Elderfield and Sherrington (1982b) that $p'(0) = \Delta_p'(0) = 0$ in order to recover the correct zero field expression ($\chi_{\rm L}$ unconstrained). The remaining constraint (51) giving the constant susceptibility $\chi_{\rm T}$(FC) in zero field apparently leads to no simple characterisation of the magnetisation, in contrast to the case of the L phase described above. Entering the mixed LT phase the character of the solution changes so that $\Delta_p'(0)$ and $p'(0)$ are non-zero, leading through (51) to the constraint

$$1 = (\beta J_{\rm L})^2 \overline{(\partial^2 f/\partial(\beta h)^2)^2} \tag{53}$$

described previously in the L phase. Consequently in *both* the mixed LT and longitudinal L spin glass phases the magnetisation profile is of the Parisi-like form

$$M/h = 1 - \tfrac{3}{4}(h^2/p(0)) + \ldots \qquad \text{longitudinal and mixed spin glasses} \tag{54}$$

where $p(0) \sim h^{2/3}$ is given by the relation (49). For the transverse T phase no simple characterisation is apparent, as illustrated by the explicit results of Elderfield and Sherrington (1982c) for the isotropic model.

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